On steady-state solutions of the coagulation equation

1996 J. Phys. A: Math. Gen. 29 1139

(http://iopscience.iop.org/0305-4470/29/5/026)

J. Phys. A: Math. Gen. **29** (1996) 1139–1140. Printed in the UK

ADDENDUM

On steady-state solutions of the coagulation equation

S Simons

School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London E1 4NS, UK

Received 30 October 1995

Abstract. Earlier work on time-independent solutions of the coagulation equation is generalized to yield explicit solutions for any kernel of the form $P(u,v) = T(u)T(v)Q(u,v)$ where Q is a homogeneous function of u and v.

The standard Smoluchowski coagulation equation is

$$\int_0^{v/2} P(u, v-u)n(u)n(v-u)\,\mathrm{d}u - n(v)\int_0^{\infty} P(u,v)n(u)\,\mathrm{d}u = \partial n/\partial t \tag{1}$$

where $n(v)$ is the number density of particles with volume v and $P(u,v)$ is the coagulation kernel. The first attempt at finding solutions of this equation which are independent of time (so that the right-hand side is zero) was made in [1] for the case of P taking the form

$$P(u,v) = \frac{T(u)T(v)}{(u+v)^3} \tag{2}$$

when it is readily shown that equation (1) has the solution

$$n(v) = \text{constant}/T(v) \tag{3}$$

for an arbitrary function $T(v)$. The original interpretation of this solution was that it corresponded to the equilibrium solution for an isolated system of coagulating particles. However, it was pointed out in [2] that such an interpretation was not possible since for an isolated system the total number of particles must be continually decreasing, thus precluding the existence of a time-independent solution. The correct interpretation was essentially given in [3] where (1) is understood to describe an 'open' system of coagulating particles connected to a reservoir of particles of zero volume which are fed into the coagulating system at a constant rate. These source particles then coagulate so that their material continually moves into particles of greater volume and the possibility then exists of setting up a steady-state situation described by (1) with the right-hand side zero. Essentially, the continual passage of material into particles of greater volume due to coagulation is balanced by the continual influx of material resulting from coagulation of the zero-volume source particles.

The purpose of the present communication is to investigate the existence of solutions to the above steady-state coagulation equation for a much wider class of kernels than those given by (2). We consider

$$P(u,v) = T(u)T(v)Q(u,v) \tag{4}$$

0305-4470/96/051139+02$19.50

where $Q(u, v)$ is a homogeneous function of u and v, satisfying

$$Q(\lambda u, \lambda v) = \lambda^{\alpha} Q(u, v) \tag{5}$$

for arbitrary constant λ and specified value of α (equation (2) is clearly a special case of this with $\alpha = -3$). We then look for a solution of (1) (with right-hand side zero) of the form

$$n(v) = B v^{\gamma} / T(v) \tag{6}$$

where B and γ are undetermined constants; we note that for $T(v) =$ constant, equation (6) corresponds to the standard Junge distribution [4]. Making use of (5) we then readily find the solution (6) satisfies the steady-state equation (1) for arbitrary value of B if γ can be found to satisfy the equation

$$\int_0^{\infty} w^{\gamma} Q(w, 1)\, \mathrm{d}w - \int_0^{1/2} w^{\gamma} (1-w)^{\gamma} Q(w, 1-w)\, \mathrm{d}w = 0\,. \tag{7}$$

We now split the integration interval $[0, \infty]$ of the first integral in (7) into two sub-intervals, $[0, 1]$ and $[1, \infty]$, and in the integral over the second sub-interval substitute $z = 1/w$. In the second integral in (7) we first use $Q(w, 1-w) = (1-w)^{\alpha} Q(w/1-w, 1)$ (since Q is a homogeneous function) and then substitute $z = w/(1-w)$. As a result, equation (7) now takes the form

$$\int_0^{1} w^{\gamma} [1 + w^{-(\alpha+2\gamma+2)} - (1+w)^{-(\alpha+2\gamma+2)}] Q(w, 1)\, \mathrm{d}w = 0 \tag{8}$$

which will hold for arbitrary Q if $-(\alpha + 2\gamma + 2) = 1$ corresponding to

$$\gamma = -\tfrac{1}{2}(\alpha + 3)\,. \tag{9}$$

It follows, therefore, that with this value of γ the solution of the steady-state coagulation equation is given by (6) for an arbitrary value of B. We note that the result obtained in [1] corresponds to a particular case of a homogeneous kernel with $\alpha = -3$.

Finally we list four cases of known kernels [5] (all with $T(v) = 1$) and the corresponding values of γ.

(i) Coagulation due to Brownian motion with particle size significantly greater than the gas molecular mean free path. Here $P(u, v) = \text{constant} \times (u^{1/3} + v^{1/3})(u^{-1/3} + v^{-1/3})$ leading to $\alpha = 0$ and $\gamma = -\frac{3}{2}$.

(ii) Coagulation due to Brownian motion with particle size significantly less than the gas molecular mean free path. Here $P(u, v) = \text{constant} \times (u^{1/3} + v^{1/3})^2 (u^{-1} + v^{-1})^{1/2}$ leading to $\alpha = \frac{1}{6}$ and $\gamma = -\frac{19}{12}$.

(iii) Coagulation due to laminar shear gas flow. Here $P(u, v) = \text{constant} \times (u^{1/3} + v^{1/3})^3$ leading to $\alpha = 1$ and $\gamma = -2$.

(iv) Coagulation due to the variation in terminal velocity of different size particles falling under gravity. Here $P(u, v) = \text{constant} \times (u^{1/3} + v^{1/3})^3 |u^{1/3} - v^{1/3}|$ leading to $\alpha = \frac{4}{3}$ and $\gamma = -\frac{13}{6}$.

References

[1] Dubovskii P B, Galkin V A and Stewart I W 1992 *J. Phys. A: Math. Gen.* **25** 4737
[2] Simons S 1993 *J. Phys. A: Math. Gen.* **26** 1259
[3] Dubovskii P B and Stewart I W 1995 *J. Phys. A: Math. Gen.* **28** 3563
[4] Twomey S 1977 *Atmospheric Aerosols* (Amsterdam: Elsevier)
[5] Friedlander S K 1977 *Smoke, Dust and Haze* (New York: Wiley)